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The Parisi–Sourlas mechanism in pseudo-Euclidean space

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Abstract. The Parisi-Sourlas mechanism is demonstrated directly for fields over pseudo-Euclidean space, without the use of Wick rotations, superspace or Berezin integration. An irreducible field supermultiplet for the Lie superalgebra iosp(m, $n|2$) is constructed using a modified version of the method of produced representations, and a (1, 1)-dimensional reduction obtained through examination of the metric.

1. Introduction

The argument of Parisi and Sourlas (Parisi and Sourlas 1979) demonstrates that a $(d+2)$-dimensional scalar field coupled to an external random source is equivalent to a free d-dimensional scalar field. The original applications of the argument were to a spin system in a random magnetic field (see, e.g., Kogon and Wallace (1981) and references therein) but quantum field theory formulations are equally possible. In order to calculate Green functions for the $(d+2)$-dimensional field, ghost fields are introduced and it is a supersymmetric invariance of the resulting Lagrangian which is responsible for the equivalence with the d-dimensional system. These supersymmetry transformations, together with the spatial rotational symmetry, the symplectic symmetry of the ghost fields and the translational and supertranslational invariance of the Green functions, form the inhomogeneous orthosymplectic Lie superalgebra iosp$(d+2|2)$, the even part of which is iso$(d+2)\oplus$sp(2). The original scalar field, the ghost fields and the random source form an iosp$(d+2|2)$ supermultiplet.

Most treatments of the Parisi-Sourlas argument (e.g., Parisi and Sourlas 1979, Cardy 1983, Klein and Perez 1983, McClain *et al* 1983, Klein *et al* 1984) introduce a superspace formalism, and show that a Berezin integral over $(d+2|2)$ superspace with an iosp-invariant integrand is equal to a d-dimensional integral over ordinary space with a similar integrand. This result is the key to showing that superfields which are invariant under an osp$(2|2)$ sub-superalgebra have Green functions identical to those of an ordinary scalar field in d dimensions.

The treatments mentioned are all for Euclidean field theories, so a Wick rotation is necessary before the procedure can be applied to relativistic field theories. Recently, there has been interest in the use of iosp$(d, 2|2)$ superfields to covariantly quantise d-dimensional gauge and string theories in the BRST formalism, making use of the Parisi-Sourlas mechanism (Siegel and Zwieback 1987, Neveu and West 1986). Unfortunately, the validity of the Wick rotation procedure here is unclear, so a pseudo-Euclidean version of the Parisi-Sourlas argument is desirable.

In this paper, an argument is given that proves the validity of the key step of the Parisi–Sourlas mechanism for a (1, 1)-dimensional reduction from an (m, n)-dimensional pseudo-Euclidean theory to an $(m-1, n-1)$-dimensional theory. The Wick rotation procedure is avoided completely. The construction is wholly algebraic for the odd part of the superalgebra and uses no supergroups, superspace coordinates or superfields, so it automatically gives irreducible field multiplets.

In § 2, an irreducible representation of the iosp$(m, n|2)$ superalgebra is constructed using the wholly algebraic method of produced representations appropriate to a classical field theory (Blattner 1969). (Here 'produced representation', 'production', etc, are used in a technical sense similar to that of induced representations (Higman 1955). The term 'co-induced' can also be found in the literature (e.g., Dixmier 1977).) As it stands, this representation cannot be used directly as a supermultiplet, so in § 3 use is made of an analogy with ordinary Poincaré field theories to modify the produced representation. The result is a unitary irreducible representation carried by a multiplet of massless fields in (m, n)-dimensional momentum space. The metric of the multiplet is examined in § 4, and the pseudo-Euclidean dimensional reduction is demonstrated.

The notation throughout this paper uses 'light-cone' coordinates for the (m, n)-dimensional space, with indices a, b, c, d taking the values $1, \ldots, m+n-2, +, -$ $(m, n > 1)$; indices λ, μ, ν, ρ the values $1, \ldots, m+n-2$; and indices for symplectic space $\alpha, \beta, \gamma, \delta = 1, 2$. Square brackets $[\cdot, \cdot]$ denote graded Lie products. The pseudo-orthogonal and symplectic metric tensors are

$$g_{\lambda\mu} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1) \qquad \text{(with -1 occuring $m-1$ times and 1 $n-1$ times)}$$

$$g_{++} = g_{--} = g_{+\lambda} = g_{\lambda +} = g_{-\lambda} = g_{\lambda -} = 0 \qquad g_{-+} = g_{+-} = 1$$

and

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

An orthosymplectic metric tensor can be defined as a square supermatrix G (with $m+n$ even rows and two odd rows) having g_{ab} and $\Omega_{\alpha\beta}$ in the diagonal partitions and zero elsewhere. The orthosymplectic superalgebra osp$(m, n|2)$ is then the set of real square supermatrices M (with $m+n$ even rows and two odd rows) satisfying

$$M^{\mathrm{st}}G + (-1)^{|M|}GM = 0$$

where st denotes the supertranspose and $|M|$ the degree of M.

2. Production of the representation

Before starting to construct the produced representation, it is well to have an explicit statement of the basis and graded Lie products of iosp$(m, n|2)$. Let J_{ab}, $K_{\alpha\beta}$, $L_{a\beta}$, P_a, Q_α be basis elements for the complexification of iosp$(m, n|2)$, with graded Lie products

$$[J_{ab}, J_{cd}] = -\mathrm{i}\hbar(g_{ac}J_{bd} - g_{bc}J_{ad} + g_{bd}J_{ac} - g_{ad}J_{bc}) \qquad (2.1a)$$

$$[K_{\alpha\beta}, K_{\gamma\delta}] = -\hbar(\Omega_{\alpha\gamma}K_{\beta\delta} + \Omega_{\beta\gamma}K_{\alpha\delta} + \Omega_{\beta\delta}K_{\alpha\gamma} + \Omega_{\alpha\delta}K_{\beta\gamma}) \qquad (2.1b)$$

$$[J_{ab}, K_{\gamma\delta}] = 0 \qquad (2.1c)$$

$$[J_{ab}, L_{c\delta}] = -\mathrm{i}\hbar(g_{ac}L_{b\delta} - g_{bc}L_{a\delta}) \qquad (2.1d)$$

$$[K_{\alpha\beta}, L_{c\delta}] = -\hbar(\Omega_{\alpha\delta}L_{c\beta} + \Omega_{\beta\delta}L_{c\alpha}) \tag{2.1e}$$

$$[L_{\alpha\beta}, L_{c\delta}] = \mathrm{i}\hbar(\Omega_{\beta\delta}J_{ac} - \mathrm{i}g_{ac}K_{\beta\delta}) \tag{2.1f}$$

$$[J_{ab}, P_c] = -\mathrm{i}\hbar(g_{ac}P_b - g_{bc}P_a) \tag{2.1g}$$

$$[K_{\alpha\beta}, Q_\gamma] = -\hbar(\Omega_{\alpha\gamma}Q_\beta + \Omega_{\beta\gamma}Q_\alpha) \tag{2.1h}$$

$$[L_{\alpha\beta}, P_c] = -\mathrm{i}\hbar g_{ac}Q_\beta \tag{2.1i}$$

$$[L_{\alpha\beta}, Q_\gamma] = -\mathrm{i}\hbar\Omega_{\beta\gamma}P_a \tag{2.1j}$$

$$[J_{ab}, Q_\gamma] = [K_{\alpha\beta}, P_c] = 0 \tag{2.1k}$$

$$[P_a, P_b] = [P_a, Q_\beta] = [Q_\alpha, Q_\beta] = 0. \tag{2.1l}$$

Comments. The basis elements J_{ab} and P_a generate $\mathrm{ISO}_0(m, n)$ rotations and translations, $K_{\alpha\beta}$ the symplectic rotations and the odd elements $L_{a\beta}$ and Q_α the super-rotations and supertranslations. The elements $(\mathrm{i}/\hbar)J_{ab}$, $(\mathrm{i}/\hbar)K_{\alpha\beta}$, $(\mathrm{e}^{\mathrm{i}\pi/4}/\hbar)L_{a\beta}$, $(\mathrm{i}/\hbar)P_a$ and $(\mathrm{e}^{\mathrm{i}\pi/4}/\hbar)Q_\alpha$ form a basis for the *real* Lie superalgebra $\mathrm{iosp}(m, n|2)$, but the given elements are convenient because they may be represented by pseudo-Hermitian operators (i.e. the even elements by Hermitian operators and the odd elements by anti-Hermitian operators).

The inhomogeneous elements P_a and Q_α span an Abelian, invariant sub-superalgebra, here denoted $\mathrm{i}(m, n|2)$, which can be used to obtain an irreducible produced representation of $\mathrm{iosp}(m, n|2)$, as follows. Let χ be the one-dimensional (irreducible) representation of $\mathrm{i}(m, n|2)$ given by

$$\chi(P_-) = 1$$

$$\chi(P_+) = \chi(P_\lambda) = \chi(Q_\alpha) = 0. \tag{2.2}$$

Note that

$$\chi(PP + QQ) = 0. \tag{2.3}$$

The little superalgebra of the homogeneous sub-superalgebra $\mathrm{osp}(m, n|2)$ for χ, consisting of all elements X of $\mathrm{osp}(m, n|2)$ for which

$$\chi([X, P_a]) = \chi([X, Q_\alpha]) = 0 \tag{2.4}$$

is isomorphic to $\mathrm{iosp}(m-1, n-1|2)$ with basis elements $J_{\lambda\mu}$, $K_{\alpha\beta}$, $L_{\lambda\beta}$, $J_{+\lambda}$, $L_{+\alpha}$, the last two sets forming a basis for the inhomogeneous part. Irreducible representations of this little superalgebra can be obtained by repeated induction or production, with finite-dimensional representations arising when the inhomogeneous part is represented trivially. However, to construct the simplest $\mathrm{iosp}(m, n|2)$ representation, the whole of the $\mathrm{iosp}(m-1, n-1|2)$ little superalgebra can be represented trivially. That is, take Δ to be the representation

$$\Delta(J_{\lambda\mu}) = \Delta(K_{\alpha\beta}) = \Delta(L_{\lambda\beta}) = \Delta(J_{+\lambda}) = \Delta(L_{+\alpha}) = 0. \tag{2.5}$$

The stability superalgebra $\mathrm{s}(\chi)$ for χ, which consists of all elements X of the whole of $\mathrm{iosp}(m, n|2)$ satisfying (2.4), is just the vector-space direct sum of the little superalgebra $\mathrm{iosp}(m-1, n-1|2)$ and the inhomogeneous part $\mathrm{i}(m, n|2)$. Let $\chi\Delta$ denote the representation of $\mathrm{s}(\chi)$ which is equal to χ for elements of $\mathrm{i}(m, n|2)$ and Δ for elements of $\mathrm{iosp}(m-1, n-1|2)$.

The elements $J_{-\lambda}$, J_{-+} and $J_{-\alpha}$ form a basis for the remainder of iosp$(m, n|2)$ (as a vector space only), and thus generate a basis for the carrier space, Z, of the universal enveloping algebra U(iosp$(m, n|2)$) regarded as a left s(χ)-module (see, e.g., Kac 1977). The basis elements of Z are of the form $\Pi_\lambda J_{-\lambda}^{r_\lambda} J_{-+}^{r_0} L_{-1}^{s_1} L_{-2}^{s_2}$, where $r_0, r_\lambda \in \mathbb{N}$ and $s_1, s_2 \in \{0, 1\}$. These can be abbreviated to

$$J^r L^s = \prod_\lambda J_{-\lambda}^{r_\lambda} J_{-+}^{r_0} L_{-1}^{s_1} L_{-2}^{s_2} \tag{2.6}$$

with $r \in \mathbb{N}^{m+n-2}$ and $s \in \{0, 1\} \times \{0, 1\}$.

With these definitions, the produced representation (Φ, V) of iosp$(m, n|2)$ can be constructed. The carrier space is the space of s(χ)-homomorphisms $V = \mathrm{Hom}_{s(\chi)}(Z, \mathbb{C})$, which is the set of linear mappings ϕ in $L(Z, \mathbb{C})$ with

$$\phi(YJ^rL^s) = \chi\Delta(Y)\phi(J^rL^s) \qquad \forall Y \in \mathrm{s}(\chi), \forall r, s. \tag{2.7}$$

Elements X of iosp$(m, n|2)$ are represented by operators $\Phi(X)$ which act on $\phi \in V$ to give $\Phi(X)\phi \in V$ defined by

$$\Phi(X)\phi(J^rL^s) = \phi(J^rL^sX) \qquad \forall r, s. \tag{2.8}$$

The construction of the simplest irreducible produced representation is thus complete. It remains only to use the graded Lie products (2.1) to reduce the term J^rL^sX to terms of the form YJ^rL^s with $Y \in \mathrm{s}(\chi)$, and then to use (2.7), (2.2) and (2.5) to complete the evaluation of (2.8). However, this procedure is impractical and the representation as it stands bears little resemblance to field theory. So the representation must be modified to an equivalent form carried by functions over coordinate or momentum space.

3. Momentum space representation

An alternative, more practical realisation of the produced representation of iosp$(m, n|2)$ is required before explicit evaluation of the linear operators can be achieved. Such a realisation can be motivated by examination of the simpler case of an ordinary Lie algebra.

Consider the Lie algebra iso(m, n) (which is an even subalgebra of iosp$(m, n|2)$). The restrictions of the representations χ and Δ to i(m, n) and so(m, n) can be used to construct an irreducible produced representation of iso(m, n) in exactly the same manner as in § 2. The little algebra of so(m, n) in this case is iso$(m-1, n-1)$, the remaining basis elements of so(m, n) being $J_{-\lambda}$ and J_{-+}, and the stability subalgebra is $\mathrm{s}_0(\chi) = \mathrm{i}(m, n) \oplus \mathrm{iso}(m-1, n-1)$. Denote the produced iso(m, n) representation by (Φ_0, V_0). The carrier space is $V_0 = \mathrm{Hom}_{s_0(\chi)}(Z_0, \mathbb{C})$, the set of $\phi_0 \in L(Z_0, \mathbb{C})$ satisfying the analogue of (2.7):

$$\phi_0(Y_0J^r) = \chi\Delta(Y_0)\phi_0(J^r) \qquad \forall Y_0 \in \mathrm{s}(\chi), r \in \mathbb{N}^{m+n-2} \tag{3.1}$$

where the basis elements of Z_0 are J^r with $r \in \mathbb{N}^{m+n-2}$. Likewise, elements X_0 of iso(m, n) are represented by operators $\Phi_0(X_0)$ with, for $\phi_0 \in V_0$,

$$\Phi_0(X_0)\phi_0(J^r) = \phi_0(J^rX_0) \qquad \forall r \in \mathbb{N}^{m+n-2}. \tag{3.2}$$

Evaluating the expressions $J'X_0$ is still very difficult. For this reason it is convenient to make use of the equivalent induced representation (Φ_0', V_0') of the Lie group $\mathrm{ISO}_0(m, n)$. The little group for $\mathrm{ISO}_0(m, n)$ is isomorphic to $\mathrm{ISO}_0(m-1, n-1)$, so the carrier space is $V_0' = C^\infty(Z_0', \mathbb{C})$, where Z_0' is the coset space of $\mathrm{ISO}_0(m, n)$ and the stability subgroup $\mathrm{S}_0(\chi) = I(m, n) \circledS \mathrm{ISO}_0(m-1,\ n-1)$, or equivalently, $\mathrm{SO}_0(m, n)/\mathrm{ISO}_0(m-1,\ n-1)$. This is isomorphic to the null surface in (m, n)-dimensional pseudo-Euclidean space, which can be parametrised by $m+n$ coordinates p^a satisfying $pp = 0$ (but not $p = 0$). For $\phi_0' \in V_0'$, the generators of this induced representation

$$\Phi_0'(J_{ab})\phi_0'(p) = \mathrm{i}\hbar\left(p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a}\right)\phi_0'(p)$$

$$\Phi_0'(P_a)\phi_0'(p) = p_a \phi_0'(p). \tag{3.3a}$$

With the definition

$$\Phi_0'(1)\phi_0'(p) = \phi_0'(p) \tag{3.3b}$$

(Φ_0', V_0') extends to a representation of the universal enveloping algebra $\mathrm{U}(\mathrm{iso}(m, n))$.

The equivalence between the induced representation (3.3) and the produced one (3.1, 3.2) for the Lie algebra is given (Blattner 1969) by assigning, for each $\phi_0' \in V_0'$, a function $\phi_0 \in L(\mathrm{U}(\mathrm{iso}(m, n)), \mathbb{C})$ defined by

$$\phi_0(A) = \Phi_0'(A)\phi_0'(k) \qquad \forall A \in \mathrm{U}(\mathrm{iso}(m, n))$$

where $k \in Z_0'$ is the stable point of the $\mathrm{ISO}_0(m-1,\ n-1)$ subgroup. It is not hard to show that in fact ϕ_0 satisfies (3.1) and thus lies in V_0. Further, if

$$\Psi_0' = \Phi_0'(X_0)\phi_0'$$

for some $X_0 \in \mathrm{iso}(m, n)$, then the function $\psi_0 \in V_0$ assigned to ψ_0' as above satisfies

$$\psi_0 = \Phi_0(X_0)\phi_0.$$

In this way, the representations (Φ_0, V_0) and (Φ_0', V_0') are essentially equivalent, so henceforth the primes will (almost) be abandoned. Which realisation is being used will be apparent from the argument of the function.

Returning to the superalgebra, there is a difficulty in simply extending the equivalence just presented to one between a produced superalgebra representation and an induced supergroup representation. An irreducible representation of a Lie superalgebra gives rise to a representation of a corresponding Lie supergroup. However, the supergroup representation may no longer be irreducible, but instead may become reducible (although perhaps not completely reducible), with the irreducible diagonal components being various induced representations of the Lie supergroup. This is so, for example, for the super-Poincaré algebra and super-Poincaré group (Williams and Cornwell 1987a, b). The relationship between Lie supergroup and superalgebra representations is thus much more complicated than that for plain Lie groups and algebras. Fortunately, for $\mathrm{iosp}(m, n|2)$ it is possible to avoid this difficulty by exploiting the correspondence just for the $\mathrm{iso}(m, n)$ subalgebra and the ordinary Lie group $\mathrm{ISO}_0(m, n)$.

Note that Z_0 is a subspace of Z, so for every $\phi \in V$ there is a function $\phi_0 \in V_0$ given by

$$\phi_0 = \phi|_{Z_0}$$

which can be expressed as a function over either Z_0 or Z'_0. Since

$$\phi(J^r L^s) = \Phi(L^s)\phi(J^r)$$

there is a one-to-one equivalence between a function $\phi \in V$ and a set of four functions in V_0 (or V'_0): ϕ_0, $(\Phi(L_{-1})\phi)_0$, $(\Phi(L_{-2})\phi)_0$, and $(\Phi(L_{-1}L_{-2})\phi)_0$. Introducing the notation

$$\phi(p) = \phi_0(p)$$
$$\phi(p, \alpha) = (\Phi(L_{-\alpha})\phi)_0(p)$$
$$\phi(p, \alpha\beta) = (\Phi(L_{-\alpha}L_{-\beta})\phi)_0(p) = -\phi(p, \beta\alpha)$$

a function $\phi \in V$ is completely determined by evaluating $\phi(p)$, $\phi(p, \alpha)$, and $\phi(p, \alpha\beta)$ for $\alpha, \beta = 1, 2$ and all p in Z'_0. So, for each $X \in \mathrm{iosp}(m, n|2)$ and $\phi \in V$, the complete specification of $\Phi(X)\phi$ (which is also a member of V) requires the calculation of $\Phi(X)\phi(p)$, $\Phi(X)\phi(p, \alpha)$, and $\Phi(X)\phi(p, \alpha\beta)$. With these definitions, it is finally possible to evaluate the action of the operators of the produced representation from their definition (2.8).

Proposition 3.1. The operators of the produced representation (Φ, V) of $\mathrm{iosp}(m, n|2)$ defined in § 2, are explicitly given in the realisation of this section by

$$\begin{aligned}
\Phi(J_{\lambda\mu})\phi(p) &= \Phi_0(J_{\lambda\mu})\phi(p)\\
\Phi(J_{\lambda\mu})\phi(p, \alpha) &= \Phi_0(J_{\lambda\mu})\phi(p, \alpha)\\
\Phi(J_{\lambda\mu})\phi(p, \alpha\beta) &= \Phi_0(J_{\lambda\mu})\phi(p, \alpha\beta)
\end{aligned} \tag{3.4a}$$

$$\begin{aligned}
\Phi(J_{-\lambda})\phi(p) &= \Phi_0(J_{-\lambda})\phi(p)\\
\Phi(J_{-\lambda})\phi(p, \alpha) &= \Phi_0(J_{-\lambda})\phi(p, \alpha)\\
\Phi(J_{-\lambda})\phi(p, \alpha\beta) &= \Phi_0(J_{-\lambda})\phi(p, \alpha\beta)
\end{aligned} \tag{3.4b}$$

$$\begin{aligned}
&\Phi(J_{+\lambda})\phi(p) = \Phi_0(J_{+\lambda})\phi(p)\\
&\Phi(J_{+\lambda})\phi(p, \alpha) = \Phi_0(J_{+\lambda})\phi(p, \alpha) + \mathrm{i}\hbar\frac{p_\lambda}{p_-}\phi(p, \alpha)\\
&\Phi(J_{+\lambda})\phi(p, \alpha\beta) = \Phi_0(J_{+\lambda})\phi(p, \alpha\beta) + 2\mathrm{i}\hbar\frac{p_\lambda}{p_-}\phi(p, \alpha\beta) + (\mathrm{i}\hbar)^2\Omega_{\alpha\beta}\Phi_0(J_{-\lambda})\phi(p)
\end{aligned} \tag{3.4c}$$

$$\begin{aligned}
\Phi(J_{-+})\phi(p) &= \Phi_0(J_{-+})\phi(p)\\
\Phi(J_{-+})\phi(p, \alpha) &= \Phi_0(J_{-+})\phi(p, \alpha) - \mathrm{i}\hbar\phi(p, \alpha)\\
\Phi(J_{-+})\phi(p, \alpha\beta) &= \Phi_0(J_{-+})\phi(p, \alpha\beta) - 2\mathrm{i}\hbar\phi(p, \alpha\beta)
\end{aligned} \tag{3.4d}$$

$$\begin{aligned}
\Phi(K_{\alpha\beta})\phi(p) &= 0\\
\Phi(K_{\alpha\beta})\phi(p, \gamma) &= \hbar(\Omega_{\alpha\gamma}\phi(p, \beta) + \Omega_{\beta\gamma}\phi(p, \alpha))\\
\Phi(K_{\alpha\beta})\phi(p, \gamma\delta) &= 0
\end{aligned} \tag{3.4e}$$

$$\begin{aligned}&\Phi(P_\lambda)\phi(p)=p_\lambda\phi(p)\\&\Phi(P_\lambda)\phi(p,\alpha)=p_\lambda\phi(p,\alpha)\\&\Phi(P_\lambda)\phi(p,\alpha\beta)=p_\lambda\phi(p,\alpha\beta)\end{aligned}\tag{3.4f}$$

$$\begin{aligned}&\Phi(P_-)\phi(p)=p_-\phi(p)\\&\Phi(P_-)\phi(p,\alpha)=p_-\phi(p,\alpha)\\&\Phi(P_-)\phi(p,\alpha\beta)=p_-\phi(p,\alpha\beta)\end{aligned}\tag{3.4g}$$

$$\begin{aligned}&\Phi(P_+)\phi(p)=p_+\phi(p)\\&\Phi(P_+)\phi(p,\alpha)=p_+\phi(p,\alpha)\\&\Phi(P_+)\phi(p,\alpha\beta)=p_+\phi(p,\alpha\beta)+(\mathrm{i}\hbar)^2\Omega_{\alpha\beta}p_-\phi(p)\end{aligned}\tag{3.4h}$$

$$\begin{aligned}&\Phi(L_{\lambda\alpha})\phi(p)=\frac{p_\lambda}{p_-}\phi(p,\alpha)\\&\Phi(L_{\lambda\alpha})\phi(p,\beta)=-\frac{p_\lambda}{p_-}\phi(p,\alpha\beta)-\mathrm{i}\hbar\Omega_{\alpha\beta}\Phi_0(J_{-\lambda})\phi(p)\\&\Phi(L_{\lambda\alpha})\phi(p,\beta\gamma)=-\mathrm{i}\hbar\Omega_{\beta\gamma}\Phi_0(J_{-\lambda})\phi(p,\alpha)\end{aligned}\tag{3.4i}$$

$$\begin{aligned}&\Phi(L_{-\alpha})\phi(p)=\phi(p,\alpha)\\&\Phi(L_{-\alpha})\phi(p,\beta)=-\phi(p,\alpha\beta)\\&\Phi(L_{-\alpha})\phi(p,\beta\gamma)=0\end{aligned}\tag{3.4j}$$

$$\begin{aligned}&\Phi(L_{+\alpha})\phi(p)=\frac{p_+}{p_-}\phi(p,\alpha)\\&\Phi(L_{+\alpha})\phi(p,\beta)=-\frac{p_+}{p_-}\phi(p,\alpha\beta)-\mathrm{i}\hbar\Omega_{\alpha\beta}\Phi_0(J_{-+})\phi(p)\\&\Phi(L_{+\alpha})\phi(p,\beta\gamma)=-\mathrm{i}\hbar\Omega_{\beta\gamma}\Phi_0(J_{-+})\phi(p,\alpha)+2(\mathrm{i}\hbar)^2\Omega_{\beta\gamma}\phi(p,\alpha)\end{aligned}\tag{3.4k}$$

$$\begin{aligned}&\Phi(Q_\alpha)\phi(p)=0\\&\Phi(Q_\alpha)\phi(p,\beta)=\mathrm{i}\hbar\Omega_{\alpha\beta}p_-\phi(p)\\&\Phi(Q_\alpha)\phi(p,\beta\gamma)=\mathrm{i}\hbar\Omega_{\beta\gamma}p_-\phi(p,\alpha).\end{aligned}\tag{3.4l}$$

Proof. Most of these transformations can be verified by straightforward application of the definition (2.8), the graded Lie products (2.1) and the property (2.7), together with the easily derived rules that for $\phi\in V$:

(i) for $X\in\mathrm{iso}(m,n)$

$$(\Phi(X)\phi)_0=\Phi_0(X)\phi_0$$

(ii) for $A, A'\in \mathrm{U}(\mathrm{iosp}(m,n|2))$

$$\Phi(A)\phi(p)=\Phi(A')\phi(p)\qquad\forall p\in Z_0'$$

if and only if

$$\Phi(A)\phi(J^r)=\Phi(A')\phi(J^r) \qquad \forall r\in\mathbb{N}^{m+n-2}$$

(iii) for $X\in \mathrm{iosp}(m, n|2)$

$$\Phi(X)\phi(p,\alpha)=\pm\Phi(X)(\Phi(L_{-\alpha})\phi)(p)+\Phi([L_{-\alpha},X])\phi(p)$$

$$\Phi(X)\phi(p,\alpha\beta)=\pm\Phi(X)(\Phi(L_{-\beta})\phi)(p,\alpha)+\Phi([L_{-\beta},X])\phi(p,\alpha)$$

the + signs applying for X even, and the − signs for X odd.

The evaluation of $\Phi(LK_{\lambda\alpha})\phi(p)$ and $\Phi(L_{+\alpha})\phi(p)$ is not so straightforward, but the two are very similar, so just $\Phi(L_{\lambda\alpha})\phi(p)$ will be shown here. Consider the action of $L_{\lambda\alpha}$ on a basis element J^r of Z_0. Using (2.6) and the commutation rule (2.1d), it is easy to see that

$$J^rL_{\lambda\alpha}=L_{\lambda\alpha}J^r+J'L_{-\alpha}$$

where $J'\in Z_0$ depends upon r and λ. Observing that $L_{\lambda\alpha}$ is an $\mathrm{so}(m-1,n-1)$ vector, just as P_λ is, it follows from (2.6) and (2.1g) that

$$J^rP_\lambda=P_\lambda J^r+J'P_-$$

with exactly the same J'. Hence, since $[L_{-\alpha},P_-]=0$

$$J^rL_{\lambda\alpha}P_- - L_{\lambda\alpha}J^rP_-=J^rP_\lambda L_{-\alpha}-P_\lambda J^rL_{-\alpha}$$

so that, using (2.7) and (2.5),

$$\Phi(L_{\lambda\alpha}P_-)\phi(J^r)=\Phi(P_\lambda L_{-\alpha})\phi(J^r).$$

Finally, using results (i) and (ii), and $[L_{\lambda\alpha},P_-]=0$,

$$\Phi(L_{\lambda\alpha})\phi(p)=(p_\lambda/p_-)\phi(p,\alpha).$$

Comments. The irreducibility of this representation is demonstrated by the evaluation of the Casimir operator. Equations (3.4f), (3.4g), (3.4h) and (3.4l) give

$$\Phi(PP+QQ)=0$$

in accord with (2.3). Note that the $\mathrm{iso}(m-1,n-1)$ subalgebra is represented 'covariantly'.

Having found an irreducible representation of $\mathrm{iosp}(m,n|2)$, an inner product is required under which the operators Φ will be pseudo-Hermitian. This is necessary so that inner products in the supermultiplet are iosp-invariant. Again, the correspondence between Lie group and Lie algebra representations provides part of the answer. An invariant measure for Z_0' is $d^{m+n}p\delta(p^2)$, so the operators Φ_0 are Hermitian under the inner product for $\phi_0,\psi_0\in V_0$ given by

$$(\phi_0,\psi_0)_0=\int \mathrm{d}^{m+n}p\,\delta(p^2)\phi_0(p)^*\psi_0(p).$$

What is required is some extension of this to Z, Φ and V. Unfortunately, the remainder of Z is not a Lie group coset space like Z_0', so a naive extension is not possible. However, all that remains to be incorporated is the extra four-dimensional space with basis elements L^s, $s\in\{0,1\}\times\{0,1\}$, so it is not too difficult to find a promising candidate.

Proposition 3.2 Under the inner product for $\phi, \psi \in V$ defined by

$$(\phi, \psi) = \int d^{m+n}p\, \delta(p^2)\Omega^{\alpha\beta}\frac{1}{p_-^2}\{\phi(p, \alpha\beta)^*\psi(p) - \phi(p)^*\psi(p, \alpha\beta) - \phi(p, \alpha)^*\psi(p, \beta) + \phi(p, \beta)^*\psi(p, \alpha)\} \tag{3.5}$$

the operators of the irreducible produced representation (3.4) of iosp($m, n|2$) are pseudo-Hermitian.

Proof. Consider first the inner product

$$(\phi, \psi)_1 = \int d^{m+n}p\, \delta(p^2)\Omega^{\alpha\beta}\{\phi(p, \alpha\beta)^*\psi(p) - \phi(p)^*\psi(p, \alpha\beta) - \phi(p, \alpha)^*\psi(p, \beta) + \phi(p, \beta)^*\psi(p, \alpha)\}. \tag{3.6}$$

Direct evaluation using the explicit expressions (3.4) shows that under (3.6) the operators $\Phi(X)$ of the irreducible representation are pseudo-Hermitian, *except* for $X = J_{+\lambda}$, J_{+-} and $L_{+\alpha}$. For these, the combinations $J_{+\lambda} - i\hbar P_\lambda P_-^{-1}$, $J_{+-} - i\hbar P_- P_-^{-1}$ and $L_{+\alpha} - i\hbar Q_\alpha P_-^{-1}$ are represented by pseudo-Hermitian operators.

The inner product (3.5) is related to (3.6) by

$$(\phi, \psi) = (\phi, \Phi(P_-)^{-2}\psi)_1$$

so the Hermiticity of $\Phi(X)$ where $[X, P_-] = 0$ is unaffected. For $J_{+\lambda}$,

$$[P_-^{-2}, J_{+\lambda}] = -2i\hbar P_\lambda P_-^{-3}$$

so, letting $\psi' = \Phi(P_-)^{-2}\psi$,

$$\begin{aligned}(\Phi(J_{+\lambda})\phi, \psi) - (\phi, \Phi(J_{+\lambda})\psi) &= (\Phi(J_{+\lambda})\phi, \psi')_1 \\ &\quad - (\phi, (\Phi(J_{+\lambda}) - 2i\hbar\Phi(P_\lambda P_-^{-1})\psi')_1 \\ &= 0\end{aligned}$$

and thus $\Phi(J_{+\lambda})$ is Hermitian under (3.5). Similar reasoning applies to J_{+-} and $L_{+\alpha}$.

Comment. While odd elements like $\Phi(Q_\alpha)$ are anti-Hermitian under (3.5), their counterparts $\Phi(Q^\alpha)$ are Hermitian. It is also worth noting that the inner product (3.5) is indefinite.

4. The dimensional reduction for the momentum representation

The key part of the dimensional reduction argument of Parisi and Sourlas consists of showing that the Green functions of the $(d+2)$-dimensional supersymmetric field theory (when the fields themselves are invariant under the osp(2|2) sub-superalgebra) are equal to those for a d-dimensional Euclidean field theory. The corresponding pseudo-Euclidean dimensional reduction can be demonstrated for a free scalar field by examining the inner product in the momentum space representation of § 3. The dimensional reduction in terms of Green functions for a quantised field theory then follows as a consequence. A limiting function with convenient properties will be used to analyse the inner product, so first this function and its properties will be examined.

Lemma 4.1. Let ε, $\eta \neq 0$, and introduce the function

$$F_{\varepsilon\eta}(p) = \frac{1}{p_+ p_- + \varepsilon^2} - \frac{1}{16\hbar^2}\left\{\frac{\varepsilon^2}{(p_+^2+\varepsilon^2)^{3/2}}\frac{1}{(p_-^2+\eta^2)^{1/2}} - \frac{\eta^2}{(p_-^2+\eta^2)^{3/2}}\frac{1}{(p_+^2+\varepsilon^2)^{1/2}}\right\}.$$

$F_{\varepsilon\eta}$ has the following properties:

(i) $$\lim_{\varepsilon,\eta\to 0} p_+ p_- F_{\varepsilon\eta}(p) = 1 \tag{4.1}$$

(ii) $$\lim_{\varepsilon,\eta\to 0} \Phi_0(J_{-+}) F_{\varepsilon\eta}(p) = \frac{-\mathrm{i}}{2\hbar}\,\delta(p_+)\delta(p_-). \tag{4.2}$$

Proof. (i) By inspection. (ii) Observe that

$$\frac{\varepsilon^2}{(p_+^2+\varepsilon^2)^{3/2}} \to 2\delta(p_+) \qquad \text{and} \qquad \frac{1}{(p_+^2+\varepsilon^2)^{1/2}} \to \frac{1}{|p_+|}$$

as $\varepsilon \to 0$, and that

$$\Phi_0(J_{-+})\frac{1}{p_+ p_- + \varepsilon^2} = 0.$$

This leaves

$$\begin{aligned}
\Phi_0(J_{-+})F_{\varepsilon\eta}(p)
&= \frac{-\mathrm{i}\hbar}{\hbar^2}\frac{\partial}{\partial p_-}(p_- F_{\varepsilon\eta}(p)) - \frac{-\mathrm{i}\hbar}{\hbar^2}\frac{\partial}{\partial p_+}(p_+ F_{\varepsilon\eta}(p))\\
&\to \frac{-\mathrm{i}\hbar}{8\hbar^2}\left\{\frac{\partial}{\partial p_-}\left(\delta(p_+)\,\mathrm{sgn}(p_-) - p_-\,\delta(p_-)\frac{1}{|p_+|}\right)\right.\\
&\qquad \left. - \frac{\partial}{\partial p_+}\left(p_+\,\delta(p_+)\frac{1}{|p_-|} - \delta(p_-)\,\mathrm{sgn}(p_+)\right)\right\}\\
&= \frac{-\mathrm{i}}{2\hbar}\,\delta(p_+)\delta(p_-)
\end{aligned}$$

since $p_-\,\delta(p_-) = 0$ and $(\partial/\partial p_-)\,\mathrm{sgn}(p_-) = 2\delta(p_-)$.

Armed with the function $F_{\varepsilon\eta}$, it is possible to prove the key component of the Parisi–Sourlas mechanism in pseudo-Euclidean space.

Proposition 4.2. Writing $\phi(p) = \phi(\boldsymbol{p}, p_+, p_-)$, the inner product (3.5) for $\phi, \psi \in V$ with

$$\Phi(L_{+\alpha})\phi = \Phi(L_{+\alpha})\psi = 0 \tag{4.3}$$

reduces to

$$(\phi, \psi) = \int \mathrm{d}^{m+n-2}\boldsymbol{p}\,\delta(\boldsymbol{p}^2)\phi(\boldsymbol{p}, 0, 0)^*\psi(\boldsymbol{p}, 0, 0).$$

Proof. If $\Phi(L_{+\alpha})\phi=0$ then, from (3.4*k*),

$$\frac{p_+}{p_-}\phi(p,\alpha)=0 \tag{4.4}$$

$$\frac{p_+}{p_-}\phi(p,\alpha\beta)+\mathrm{i}\hbar\Omega_{\alpha\beta}\Phi_0(J_{-+})\phi(p)=0. \tag{4.5}$$

Since $\phi(p,\alpha)\in C^\infty(Z_0',\mathbb{C})$ for α fixed, the only solution to (4.4) is

$$\phi(p,\alpha)=0.$$

Similar equations apply to ψ, leaving the inner product as

$$(\phi,\psi)=\int \mathrm{d}^{m+n}p\,\delta(p^2)\Omega^{\alpha\beta}\frac{1}{p_-^2}\{\phi(p,\alpha\beta)^*\psi(p)-\phi(p)^*\psi(p,\alpha\beta)\}.$$

Introducing $p_+p_-F_{\varepsilon\eta}(p)$ into the inner product by (4.1),

$$(\phi,\psi)=\lim_{\varepsilon,\eta\to0}(\phi,p_+p_-F_{\varepsilon\eta}\psi).$$

Evaluating the right-hand side, making use of (4.5) and integrating by parts,

$$(\phi,p_+p_-F_{\varepsilon\eta}\psi)=\int \mathrm{d}^{m+n}p\,\delta(p^2)\phi(p)^*\psi(p)2\mathrm{i}\hbar\Phi_0(J_{-+})F_{\varepsilon\eta}(p)$$

$$\to\int \mathrm{d}^{m+n}p\,\delta(p^2)\phi(p)^*\psi(p)\delta(p_+)\delta(p_-)$$

as $\varepsilon,\eta\to0$, by (4.2). Integrating out p_+ and p_- gives the desired result.

Comments. The required reduction of one space and one time dimension has been achieved, together with the elimination of the parts of the supermultiplet not contained in V_0. What remains is the standard inner product for the momentum representation of an ordinary classical massless scalar field carrying a unitary irreducible representation of iso$(m-1,n-1)$. The condition (4.3) is less restrictive than the osp(2|2) invariance used in the usual treatments.

The dimensional reduction for the Green functions of quantised fields follows by virtue of the direct correspondence between vacuum expectation values for quantised fields and the inner product for classical fields. For example, for the ordinary $(m-1,n-1)$-dimensional field, a general state $|\phi\rangle$ of the quantum field can be written in terms of the reaction operators $a(\boldsymbol{p})^\dagger$ as

$$|\phi\rangle=\int \mathrm{d}^{m+n-2}\boldsymbol{p}\,\delta(\boldsymbol{p}^2)\phi(\boldsymbol{p})a(\boldsymbol{p})^\dagger|\rangle$$

where $|\ \rangle$ is the vacuum state. The inner product for such general states can be defined to be the same as that for the coefficient functions considered as classical fields,

$$\langle\phi|\psi\rangle=\int \mathrm{d}^{m+n-2}\boldsymbol{p}\,\delta(\boldsymbol{p}^2)\phi(\boldsymbol{p})^*\psi(\boldsymbol{p}).$$

The commutation rules for the annihilation and creation operators, or equivalently, the momentum space Green functions then follow:

$$\langle|a(\boldsymbol{p})a(\boldsymbol{p}')^\dagger|\rangle=2p_1\,\delta(p_2-p_2')\ldots\delta(p_{m+n-2}-p'_{m+n-2}).$$

Coordinate space Green functions can be obtained by Fourier transformation.

5. Summary

The kernel of the Parisi–Sourlas mechanism in pseudo-Euclidean space has been demonstrated in a direct manner without recourse to Wick rotations. A unitary irreducible iosp$(m, n|2)$ supermultiplet of classical momentum space fields has been constructed using a modified version of the method of produced representations. This procedure has the advantage of being wholly algebraic for the odd part of the superalgebra, and thus avoids the need for superspace, superfields or supergroups. The required (1,1)-dimensional reduction for the (m, n)-dimensional fields appears through examining the metric of the supermultiplet. Extensions of the argument to coordinate space, quantum fields, higher spin fields, and so on then follow in the same manner as for Euclidean fields.

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